A regularized approach for solving magnetic differential equations and a revised iterative equilibrium algorithm

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A method for approximately solving magnetic differential equations is described. The approach is to include a small diffusion term to the equation, which regularizes the linear operator to be inverted. The extra term allows a "source-correction" term to be defined, which is generally required in order to satisfy the solvability conditions. The approach is described in the context of computing the pressure and parallel currents in the iterative approach for computing magnetohydrodynamic equilibria. © *2010 American Institute of Physics*. [doi:10.1063/1.3506821]

A magnetic differential equation[1] is an equation of the form

$$\mathbf{B}\cdot\nabla f = s, \tag{1}$$

where $\mathbf{B}$ is a given magnetic field, s is an arbitrarily prescribed source term, and f is a single valued function that is to be determined. This equation describes how f changes while traversing along a field line. Toroidal magnetic fields are analogous to $1\frac{1}{2}$ dimensional Hamiltonian systems[2] and consequently, in the absence of a continuous symmetry, are generally chaotic.[3] This paper shall suggest numerically tractable techniques for solving this equation to arbitrary accuracy.

Equations of this type arise in several contexts in plasma physics. The electric field that satisfies Ohm's law, $\mathbf{E}+\mathbf{v}\times\mathbf{B}=\eta\mathbf{j}$, for a given steady-state magnetic field, so that $\mathbf{E}=-\nabla\phi$, is given by $\mathbf{B}\cdot\nabla\phi=-\eta\mathbf{j}\cdot\mathbf{B}$. In the solution to the ideal magnetohydrodynamic (MHD) equilibrium equation, $\nabla p=\mathbf{j}\times\mathbf{B}$, magnetic differential equations arise for the pressure, $\mathbf{B}\cdot\nabla p=0$, and the parallel current, $\mathbf{B}\cdot\nabla\sigma=-\nabla\cdot(\mathbf{B}\times\nabla p/B^2)$, where $\sigma=\mathbf{j}\cdot\mathbf{B}/B^2$ and we have assumed $\nabla\cdot\mathbf{j}=0$. We will discuss these latter two equations in more detail below.

Several problems arise when solving equations of this type. The first problem is that, in toroidal geometry, if f is to be single valued then the source term must satisfy certain solvability conditions:[4] if one were to integrate Eq. (1) along a closed field line (i.e., a periodic orbit), we must have $\oint s dl/B=0$. This condition generally will *not* be satisfied for arbitrary s. A robust method for solving Eq. (1) should not be sensitive to small violations of the solvability conditions. In the approach adopted below, a "source-correction" term δs will be computed so that $s+\delta s$ *does* satisfy the solvability conditions.

The second, related problem is that the $\mathbf{B}\cdot\nabla$ is *pathologically* singular when the field is chaotic. Magnetic differential equations are singular on periodic orbits, and periodic orbits densely populate chaotic fields. One could imagine an approach that explicitly located periodic orbits, and either included a term to cancel $\oint s dl/B=0$, or avoided the periodic orbits altogether. However, it is not always easy to locate periodic orbits in chaotic fields, and regularizing and inverting the $\mathbf{B}\cdot\nabla$ operator by this approach is tantamount to resolving the infinitely complicated, fractal structure of the magnetic field.[5] This is no easy task, and is unlikely to lead to a reliable numerical algorithm.

The third problem is that the solution is arbitrary to within a function that is constant along a field line: if f is a solution then so is $\bar{f}=f+\alpha$ for any α that satisfies $\mathbf{B}\cdot\nabla\alpha=0$. In the case of integrable fields, where each field line lies on a toroidal surface (i.e., a flux-surface) and the flux-surfaces themselves are continuously nested, the integration constant may be an arbitrary surface function, $\alpha=\alpha(s)$ where s labels flux-surfaces. In the chaotic regions however, where the field lines associated with the unstable manifolds of the unstable periodic orbits seem to wander about randomly, and where there may exist some surviving irrational flux-surfaces [so-called Kolmogorov, Arnold, and Moser[3] (KAM) surfaces] and cantori,[6] and where there may exist small island chains about the stable periodic orbits,[7] it is not at all obvious how one can choose a nontrivial integration function so that f will be continuous.

The approach adopted here is to include a nonsingular, linear operator to the left hand side of Eq. (1). Consider the advection-diffusion equation

$$\mathbf{B}\cdot\nabla f + D\nabla\cdot\nabla_\perp f = s, \tag{2}$$

where D is assumed to be a small constant and $\nabla_\perp f \equiv \nabla f - \mathbf{bb}\cdot\nabla f$. Provided $D\neq 0$, the operator $\mathcal{L}\equiv\mathbf{B}\cdot\nabla + D\nabla\cdot\nabla_\perp$ is nonsingular and $\mathcal{L}f=s$ is readily inverted. An intuitive understanding of why this is so is to note that the $\nabla\cdot\nabla_\perp$ operator contains higher order derivatives than $\mathbf{B}\cdot\nabla$. Wherever f tends to be singular, the higher order derivatives, and thus the regular diffusion process, will dominate.

As with all differential equations, to obtain a unique solution we must supplement Eq. (2) with boundary conditions. Integrating Eq. (2) over some volume V, with boundary ∂V, we obtain

$$\oint_{\partial V}(f\mathbf{B}+D\nabla_\perp f)\cdot d\mathbf{a} = \int_V s dv, \tag{3}$$

where $d\mathbf{a}$ is the normal area element. Suitable boundary conditions are problem dependent.

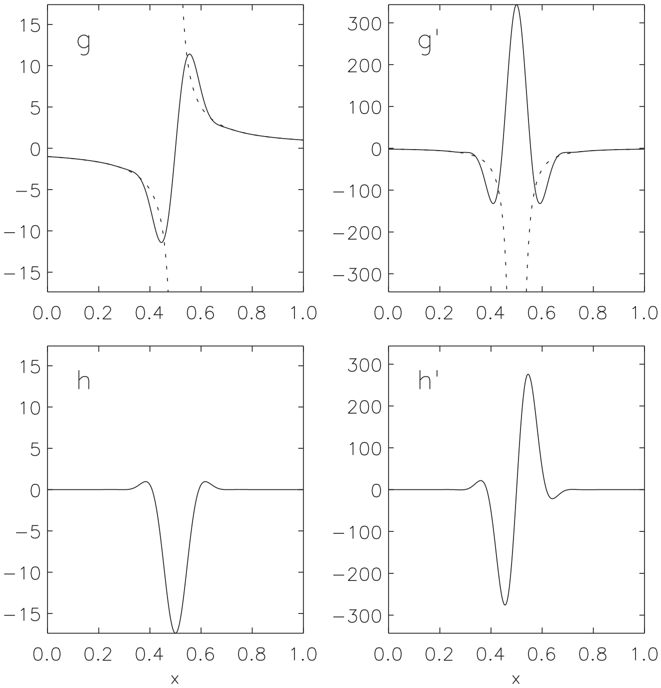

FIG. 1. Solutions, $g(x)$ and $h(x)$, and their derivatives for Eq. (2) with $D=10^{-4}$.

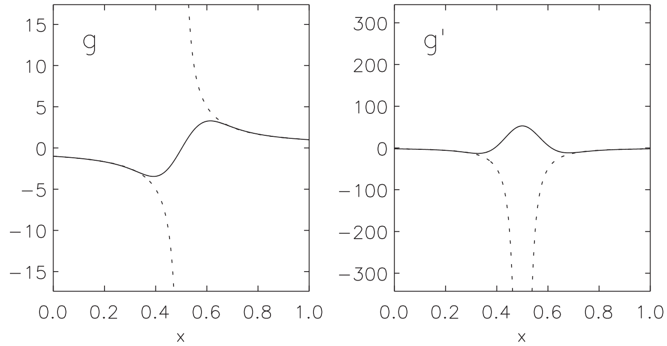

FIG. 2. Solution $g(x)$ and its derivative for Eq. (4) with $D=10^{-4}$.

After computing the solution to Eq. (2), we may identify the source-correction term, $\delta s = -D\nabla\cdot\nabla_\perp f$, and we have effectively solved $\mathbf{B}\cdot\nabla f = s+\delta s$. By taking the limit $D\to 0$, and assuming that $\nabla\cdot\nabla_\perp f$ remains finite, we uncover the solution $\mathbf{B}\cdot\nabla f = s$.

Note that the solutions to Eq. (1) are not required to be smooth. Where the differential equation is singular the mathematical solution admits a δ-function singularity. In contrast, the solutions to Eq. (2), and also to Eq. (4) below, are guaranteed to be continuous and smooth for nonzero D. So, the regularization approach suggested here automatically approximates the continuous solution to Eq. (1), which we assume is the physically relevant solution in the context of MHD calculations.

To illustrate, consider a magnetic field in Cartesian coordinates, $\mathbf{B}_0=\nabla\times(x\nabla y-\chi_0\nabla z)$, where x serves as the radial coordinate, and y, z are the poloidal, toroidal angles. We choose a simple integrable field, $\chi_0(x)=x^2/2$, to give a linear transform profile, $\iota\equiv\chi_0'=x$. For the source term we use $s=s_{mn}(x)\cos(my-nz)$. The solution to Eq. (1) is $\bar{g}(x)\sin(my-nz)$, where $\bar{g}=s_{mn}/(\iota m-n)$, which is singular at $\iota=n/m$. This solution and its derivative are shown as the dotted curves in Figs. 1 and 2.

We write the solution to Eq. (2) as a sum of odd and even components, $f=g(x)\sin(my-nz)+h(x)\cos(my-nz)$, and obtain a coupled pair of second-order, differential equations for g and h. To be explicit, we use $m=2$ and $n=1$ and take $s_{mn}(x)=1$. A finite difference method is used, with the boundary condition that $g=\bar{g}$ and $h=0$ at $x=0$ and at $x=1$. The solutions, g and h, and their derivatives, are shown in Fig. 1.

There is a subtlety here that needs to be recognized. If the original source term satisfies the solvability conditions, then the solution is well behaved and no source-correction term is required. In this case, the term δs vanishes as D approaches zero. If the source *does not* satisfy the solvability conditions, then as D becomes smaller the solution approaches a singular/pathological limit and ∇f grows without bound. In this case, δs remains finite even as $D\to 0$, but it is, in some sense, no larger than required and localized to where the solvability constraint is violated.

For more complicated geometry and fields, Eq. (2) may be solved by a variety of numerical methods. For example, a locally field aligned coordinate grid will accurately resolve the $\mathbf{B}\cdot\nabla$ operator.[8] We make no assumption regarding the structure of the field, and the approach is equally valid for integrable magnetic fields, partially chaotic fields, and fields that are so chaotic they are effectively random. We do not need to resolve the fractal structure of the chaotic field; it is only required to resolve the structure of the solution, and provided D is nonzero, the solution will be smooth.

Magnetic differential equations arise in the iterative solution of the MHD equilibrium equations.[9] The ideal force balance equation is $\mathbf{j}\times\mathbf{B}=\nabla p$. The iterations proceed by identifying $\mathbf{j}_\perp=\mathbf{B}\times\nabla p/B^2$. Writing $\mathbf{j}=\sigma\mathbf{B}+\mathbf{j}_\perp$ and insisting that $\nabla\cdot\mathbf{j}=0$ we obtain a magnetic differential equation for the parallel current, $\mathbf{B}\cdot\nabla\sigma=-\nabla\cdot(\mathbf{B}\times\nabla p/B^2)$. Assuming this equation can be solved, the magnetic field is updated by inverting $\nabla\times\mathbf{B}=\mathbf{j}$. To be consistent with an ideal equilibrium, the pressure is adjusted in order to satisfy $\mathbf{B}\cdot\nabla p=0$. This closes the Picard-style iterative loop. Further details are given in Refs. 10 and 11, and some effort has been devoted to implementing this approach computationally.[12]

However, this iterative scheme depends on solving two magnetic differential equations. These equations are singular, and the difficulties encountered in inverting these equations described earlier result in such an algorithm being numerically ill-posed. This is related to the fact that the solutions to $\nabla p=\mathbf{j}\times\mathbf{B}$ are ill-defined when the fields are chaotic. The only continuous pressure that is consistent with $\mathbf{B}\cdot\nabla p=0$ for chaotic fields is something akin to a devil's staircase:[5] the pressure-gradient is either discontinuous or zero. The same is true for the perpendicular current, $\mathbf{j}_\perp\equiv\mathbf{B}\times\nabla p/B^2$, which does not have a well defined, nontrivial divergence, and $\mathbf{B}\cdot\nabla\sigma=-\nabla\cdot\mathbf{j}_\perp$ cannot be solved.

We shall slightly modify this iterative scheme in order to remove the problems associated with inverting these two equations, and thus obtain an algorithm that is both numerically tractable and transparent, and that will hopefully allow

MHD equilibria to be constructed faithfully even for arbitrarily chaotic fields. Consider first the solution to $\mathbf{B}\cdot\nabla p=0$. The solvability conditions for this equation are trivially satisfied; however, for chaotic fields (which have structure on all length scales) the only nontrivial, continuous solutions have an uncountable infinity of discontinuities in ∇p.[5] This pathological structure can be removed by introducing small, nonideal terms.

Recognizing that realistically the transport of the pressure along the magnetic field is not infinitely fast, that the pressure will relax *in both* directions along the magnetic field and so the parallel transport should be modeled by a second-order differential equation, and that particle collisions, finite-Larmor radius effects, etc. will result in some small perpendicular transport, a reasonable alternative to $\mathbf{B}\cdot\nabla p=0$ is the anisotropic-diffusion equation

$$\nabla\cdot(\nabla_{\parallel}p + D\nabla_{\perp}p) = Q, \tag{4}$$

where $\nabla_{\parallel}p \equiv \mathbf{bb}\cdot\nabla p$ and $\nabla_{\perp}p\equiv\nabla p-\nabla_{\parallel}p$, and Q is a source term which may be used to drive nontrivial solutions. Nontrivial solutions may also be forced by inhomogeneous boundary conditions. As D becomes smaller, the pressure adapts mores closely to any flux-surfaces that may exist, and even to the cantori,[8] but for nonzero D magnetic islands smaller than a critical island width, $\Delta w=\mathcal{O}(D)^{1/4}$, are inconsequential,[13] and the pressure is smooth, even for chaotic fields.

Solutions to the anisotropic-diffusion equation may serve as approximate solutions to magnetic differential equations. Shown in Fig. 2 are the solutions to Eq. (4) for the integrable field considered earlier. To approximate the solutions to Eq. (1) using Eq. (4), we must take $Q=\mathbf{B}\cdot\nabla(s/B^2)$, and it is sufficient to write $f=g(x)\sin(my-nz)$.

In fact, for slightly perturbed fields, Eq. (1) can be transformed to an equation that is similar in form to Eq. (4). Consider a magnetic field given by $\mathbf{B}=\mathbf{B}_0+\delta\mathbf{B}$. The outstanding effect of the perturbation is to introduce a radial derivative, so that $\mathbf{B}\cdot\nabla\approx\mathbf{B}_0\cdot\nabla+\delta B^x\partial_x$. Now consider applying this operator once more to Eq. (1)

$$(\mathbf{B}\cdot\nabla)(\mathbf{B}\cdot\nabla f) = \mathbf{B}\cdot\nabla s. \tag{5}$$

Realizing that for $\delta=0$ the solution is singular, and so the highest order radial derivative of f will dominate, and by discarding small terms, this may be approximated by

$$(\mathbf{B}_0\cdot\nabla)(\mathbf{B}_0\cdot\nabla)f + (\delta B^x)^2\partial^2_{xx}f = \mathbf{B}_0\cdot\nabla s \tag{6}$$

which, after a suitable averaging operation, is essentially an anisotropic-diffusion equation, and is nonsingular. While this is not a rigorous proof, it suggests that the effect of a small radial field, $(\delta B^x)^2\partial^2_{xx}f$, is to induce a small diffusion of f across the unperturbed flux-surfaces. If one assumes that the perturbed field lines are stochastic, and then assumes that the degree of stochasticity is sufficient so that the radial deviation of the field lines is effectively random, a quasilinear expression for the radial diffusion coefficient could be derived.[14,15] If one replaces the radial diffusion term, $(\delta B^x)^2\partial^2_{xx}f$, with $-\eta^2 f$, then the usual resonance broadening heuristic is obtained, $f_{m,n}=(\iota m-n)s_{m,n}/[(\iota m-n)^2+\eta^2]$, where we have employed a Fourier representation for f and s, and used $\mathbf{B}_0\cdot\nabla\equiv(\iota m-n)$.

Including a perturbed radial field *appears* to have eliminated the singularity, however, Eq. (5) is no less and no more singular than Eq. (1). The resonance broadening described by Eq. (6) has been achieved by somewhat vaguely discarding small terms (arguing that the perturbation terms are small and that the field is almost integrable) and/or by averaging (arguing that the perturbation is sufficient to cause strong field line chaos). This is equivalent to altering the linear operator that acts upon f. It is not obvious how well solutions to Eq. (6) will approximate solutions to Eq. (1) for arbitrarily chaotic fields. The singularities in the $\mathbf{B}\cdot\nabla$ have *not* been removed by the introduction of perturbed radial or chaotic fields. The singularities are associated with the existence of periodic orbits, and periodic orbits are guaranteed to survive perturbation (for any system with shear) by the Poincaré–Birkhoff theorem.[3] Magnetic differential equations are *guaranteed* to be singular for toroidal magnetic fields, regardless of the degree of chaos.

From a numerical perspective, Eq. (4) has several advantages over Eq. (6). Equation (6) would not regularize the singularities when the field is integrable, whereas the resonance broadening in Eq. (4) is explicit and transparent, and independent of the degree of chaos. To exploit Eq. (6) numerically it is required to represent the given field as a small perturbation to a "nearby"-integrable field, as this determines the magnitude of δ, but there is some arbitrariness in the choice of nearby-integrable field across which f is assumed to be diffusing: if the nearby-integrable field is chosen poorly, a rather peculiar diffusion process could result. It would seem that the best method of ensuring that the diffusion perpendicular to the coordinate surfaces was consistent with a small diffusion perpendicular to the field would be to use coordinates adapted to the surviving invariant magnetic surfaces and cantori, i.e., chaotic-coordinates.[16] Indeed, the construction of chaotic-coordinates allows Eq. (4) to be solved analytically: assuming that p takes the form $p=p(\psi)$, where ψ labels the coordinate surfaces, and ignoring the source for simplicity, the solution to Eq. (4) is given by $p'\propto(\varphi+DG)^{-1}$, where φ is the squared field-line flux across the coordinate surfaces and G is an average metric quantity.[16]

The second magnetic differential equation that arises is for the parallel current. Either to ensure that the solvability conditions are satisfied, or to allow small pressure gradients along the magnetic field, which would be the case if Eq. (4) is used, we must include additional terms to the force balance equation.[17] Generally, we can write

$$\mathbf{j}\times\mathbf{B} = \nabla p + \mathbf{u}_{\perp}\times\mathbf{B} + \lambda\mathbf{B}. \tag{7}$$

The $\lambda\mathbf{B}$ term in Eq. (7) does not contribute to perpendicular force balance. If the pressure satisfies Eq. (4), then this term is small, $\lambda\sim D$. The parallel current is given by the magnetic differential equation

$$\mathbf{B}\cdot\nabla\sigma = -\nabla\cdot(\mathbf{B}\times\nabla p/B^2) - \nabla\cdot\mathbf{u}_{\perp}. \tag{8}$$

We may solve this, and compute the additional perpendicular current $\mathbf{u}_{\perp}$ by solving instead the regularized equation

$$\mathbf{B} \cdot \nabla\sigma + D \nabla \cdot \nabla_{\perp}\sigma = -\nabla \cdot (\mathbf{B} \times \nabla p/B^2) \tag{9}$$

and then identify $D\nabla\cdot\nabla_{\perp}\sigma=\nabla\cdot\mathbf{u}_{\perp}$. This may be integrated to obtain $\mathbf{u}_{\perp}=D\nabla_{\perp}\sigma$. There is an arbitrary integration term, $\nabla\times h$, but this may be set to zero. The operator that acts on σ in Eq. (9) is no longer singular, and this removes the δ-function solution to the parallel current that would otherwise generally be present.[18]

As for the boundary conditions that are required to supplement Eq. (9), consider for example a region bounded by two irrational flux-surfaces (so-called KAM surfaces), on which $\mathbf{B}\cdot\mathbf{n}=0$, where $\mathbf{n}$ is normal to the boundary surfaces. It is sufficient to supply Dirichlet boundary conditions, and σ on the computational boundary may be specified arbitrarily. A natural choice is constructed as follows. On the irrational surfaces the operator $\mathbf{B}\cdot\nabla$ is nonsingular, and $\mathbf{B}\cdot\nabla\sigma=-\nabla\cdot(\mathbf{B}\times\nabla p/B^2)$ is readily inverted to solve for σ, for example by constructing straight-field line coordinates.[19,20] The boundary condition may be taken as this solution plus any constant.

We have presented a method that (i) allows the solutions of magnetic differential equations to be approximated arbitrarily closely; (ii) self-consistently provides a small, source-correction term where it is required; and (iii) can be employed for general magnetic fields without making any assumptions regarding the chaotic structure of the field.

The $D\nabla\cdot\nabla_{\perp}$ term is a smoothing operator, with a scale length controlled by D. This term resolves the singularities that would otherwise occur. Though the inclusion of this term is somewhat arbitrary, the resultant source-correction term should not be thought of as artificial. For example, if $\mathbf{j}_{\perp}=\mathbf{B}\times\nabla p/B^2$ is *not* consistent with $\nabla\cdot\mathbf{j}=0$, then there *must* be some additional force that drives a perpendicular current. To the extent that this additional force is small and localized, the final result is likely to be somewhat insensitive to the precise details, as long as the singular structure is removed.

Any number of smoothing operators could be added, and the form of the operator will determine the form of the additional force. Conceivably, one could choose an operator that mimics the effect of a small plasma velocity, for example. As shown in Fig. 1, the advection-diffusion equation couples functions of different symmetry: while this may be acceptable for modeling non up-down symmetric devices, the additional force in this case would violate stellarator symmetry. The anisotropic-diffusion equation in contrast preserves any symmetry that may be present.

As D becomes smaller and the regularization term becomes weaker, the parallel currents etc. will become increasingly localized and so the numerical resolution required to resolve these structures will increase. An operator with higher derivatives would have the effect of further localizing, for a given D, the source-correction term to regions where the solvability constraint was violated, but again this would come at the expense of requiring enhanced numerical resolution to resolve the increasingly localized structures. In any case, provided D is small, the source correction term arising from the $D\nabla\cdot\nabla_{\perp}$ term will be both small and localized.

We have suggested a modified iterative procedure for calculating MHD equilibrium solutions. By regularizing the magnetic differential equations arising for the pressure and parallel current, the fractal, singular structure of the equilibrium solutions is removed. By taking D to be small, we hope in future to compute a nontrivial, nearly ideal equilibrium with arbitrarily chaotic fields.

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